Lecture notes on Exact Differential equation

Prasad T.Bathe
Assistant Professor in Mathematics,
Rayat Shikshan Sanstha's
Mahatma Phule Mahavidyalaya, Pimpri, Pune-411017

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1 Transformation of Nonlinear Equations into Separable Equations

Consider a linear non homogeneous equation

$$y' + p(x)y = f(x) \tag{1}$$

whose solution is of the form

$$y = uy_1 \tag{2}$$

where y_1 is a nontrivial solution of

$$y' + p(x)y = 0 \tag{3}$$

and u is a solution of

$$u' = \frac{f(x)}{y_1(x)} \tag{4}$$

In this section we will consider nonlinear differential equations that are separable to begin with, but can be solved by writing their solutions in the form $y = uy_1$, where y_1 is a suitably chosen known function and u satisfies a separable equation.

1.1 Bernoulli Equation

A Bernoulli's Equation is an equation of the form

$$y' + p(x)y = f(x)y^n \tag{5}$$

where n can be any real number other than 0 or 1.

1.1.1 Solution of Bernoulli Equation

General solution of equation (5) is given by

$$\int \frac{du}{u^n} = \int f(x) \left(y_1(x)\right)^{n-1} dx \tag{6}$$

where y_1 is a nontrivial solution of

$$y' + p(x)y = 0$$

and substitute

$$u = \frac{y}{y_1}$$

in solution of equation (6) to get the required solution Bernoulli Equation.

1.2 Homogeneous Nonlinear Equations

The nonlinear first order differential equation

$$y' = f(x, y) \tag{7}$$

is said to be homogeneous if x and y occur in f in such a way that $f(x, y)$ depends only on the ratio $\frac{y}{x}$

2 Exact Equations

The first order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0 \tag{8}$$

is said to be exact if there exists a function $u(x, y) = c$ such that,

$$du = M(x, y)dx + N(x, y)dy$$

2.0.1 Necessary and Sufficient Condition for Exact

The first order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

3 Integrating Factors

A function $\mu(x, y)$ is an integrating factor for the first order differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

if

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact.

3.0.1 Rules for Finding Integrating Factor

- If $\frac{(M_y - N_x)}{N}$ is independent of y , that is $\frac{(M_y - N_x)}{N} = p(x)$ then the integrating factor is given by

$$\mu = e^{\int p(x)dx}$$

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$\frac{f'(c)}{g'(c)} = \frac{f(1) - f(0)}{g(1) - g(0)}$ therefore

$$\frac{3c^2}{\frac{1}{1 + c^2}} = \frac{1}{\pi/4}$$